Solution to MM Problem # 1829

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Problem# 1829 *Proposed by Oleh Faynshteyn, Leipzig, Germany.* Let ABC be a triangle with $BC = a$, $CA = b$, and $AB = c$. Let r_a denote the radius of the excircle tangent to BC, r_b the radius of the excircle tangent to CA, and r_c the radius of the excircle tangent to AB. Prove that

$$\frac{r_a r_b}{(a+b)^2} + \frac{r_b r_c}{(b+c)^2} + \frac{r_c r_a}{(c+a)^2} \geq \frac{9}{16}$$

Solution. Let s and S be respectively the semi-perimeter and area of triangle ABC. It is well known that $r_a = \dfrac{S}{s-a}$, $r_b = \dfrac{S}{s-b}$, $r_c = \dfrac{S}{s-c}$, and $S^2 = s(s-a)(s-b)(s-c)$. Using these relations, we readily simplify

$$\frac{(s-c)}{(a+b)^2} + \frac{(s-a)}{(b+c)^2} + \frac{(s-b)}{(c+a)^2} \quad \geq \quad \frac{9}{16s}$$

Letting $a = (y+z)/2$, $b = (x+z)/2$, and $c = (x+y)/2$ we obtain

$$\frac{2z}{(x+y+2z)^2} + \frac{2x}{(2x+y+z)^2} + \frac{2y}{(x+2y+z)^2} \quad \geq \quad \frac{9}{8(x+y+z)}$$

Clearing the denominators and simplyfing we get

$$\begin{aligned} & 210(x^3y^2z + x^3yz^2 + x^2y^3z + xy^2z^3 + x^2yz^3 + xy^3z^2) \\ & +666x^2y^2z^2 \\ \geq\ & 20(x^6+y^6+z^6) + 76(x^5y + x^5z + y^5z + yz^5 + xy^5 + xz^5) + \\ & 133(x^4z^2 + x^2y^4 + x^4y^2 + x^2z^4 + y^4z^2 + y^2z^4) + 50(x^4yz + xy^4z + xyz^4) + \\ & 154(x^3y^3 + x^3z^3 + y^3z^3) \end{aligned}$$

Now, using the notation $[\alpha, \beta, \gamma] = \sum_{\text{sym}} x^\alpha y^\beta z^\gamma$, previous expression may be written as:

$$210[3,2,1] + 111[2,2,2] \geq 10[6,0,0] + 76[5,1,0] + 133[4,2,0] + 25[4,1,1] + 77[3,3,0]$$

Each sequence into the brackets of the right-hand side majorises every sequence into the brackets of the left-hand side of the last inequality. For example $(3,3,0) \succ (3,2,1)$. Therefore the **reversed inequality**

$$\frac{r_a r_b}{(a+b)^2} + \frac{r_b r_c}{(b+c)^2} + \frac{r_c r_a}{(c+a)^2} \leq \frac{9}{16}$$

holds by Muirheads inequality. Finally, note that since the sum of the coefficients is the same in both sides, the equality holds if and only if $x = y = z$ that is when the triangle is equilateral. $\square$

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