

(1)

$$\textcircled{1} \quad y^2 z_x + x y z_y = x \quad \text{con} \quad z = t - y^2$$

sobre la curva  $x = 0$

$$\frac{dx}{y^2} = \frac{dy}{xy} = \frac{dz}{x}$$

$\underbrace{\phantom{...}}$

$$x dx = y dy$$

$$\boxed{x^2 - y^2 = u_1}$$

$$\frac{dy}{y} = dz$$

$$luy = z + d$$

$$\boxed{u_2 = -z + luy}$$

$$\left. \begin{array}{l} x = 0 \\ y = t \\ z = 1 - t^2 \end{array} \right\}$$

$$u_1 = -t^2$$

$$\longrightarrow t = \sqrt{-u_1}$$

$$u_2 = -1 + t^2 + \ln t$$

$$u_2 = -1 - u_1 + \ln \sqrt{-u_1}$$

$$\text{e.d. la soluci\'on es } -z + luy = -1 - x^2 + y^2 + \ln \sqrt{y^2 - x^2}$$

$$\text{e.d. } z = luy - \ln \sqrt{y^2 - x^2} + x^2 + y^2 + 1$$

$$\boxed{z = \ln \frac{y}{\sqrt{y^2 - x^2}} + x^2 + y^2 + 1}$$

$$\textcircled{2} \quad u_t = u_{xx} \quad 0 < x < 1, t > 0$$

$$u_x(0,t) = u(1,t) = 0, t > 0$$

$$u(x,0) = 1 - x^2 ; \quad 0 < x < 1$$

$$-\left(\frac{2n+1}{2}\pi\right)^2 t$$

$$\frac{T'}{T} = \frac{x''}{x} = -\lambda^2 \rightarrow T_n(t) = e^{-\left(\frac{2n+1}{2}\pi\right)^2 t}$$

$$X = A \cos \lambda x + B \sin \lambda x$$

$$u(1,t) = 0 \Rightarrow A \cos \lambda + B \sin \lambda = 0$$

$$u_x(0,t) = 0 \Rightarrow -\lambda A \sin \lambda 0 + \lambda B \cos \lambda 0 = 0 \quad \begin{cases} \lambda = 0 \\ B = 0 \end{cases}$$

$$B=0 \Rightarrow A \cos \lambda = 0 \rightarrow \omega \lambda = 0 \rightarrow \boxed{\lambda = \frac{\pi}{2} + n\pi} \quad n = 0, 1, 2, \dots$$

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-\left(\frac{2n+1}{2}\pi\right)^2 t} \cdot \cos\left(\frac{2n+1}{2}\pi x\right)$$

$$u(x,0) = 1 - x^2 = \sum A_n \cos\left(\frac{2n+1}{2}\pi x\right)$$

$$\Rightarrow A_n = \frac{2}{1} \int_0^1 (1 - x^2) \cos\left(\frac{2n+1}{2}\pi x\right) dx$$

$$u = 1 - x^2 \quad du = -2x dx$$

$$dV = \cos\left(\frac{2n+1}{2}\pi x\right) dx \rightarrow V = \frac{\sin\left(\frac{2n+1}{2}\pi x\right)}{\frac{2n+1}{2}\pi}$$

$$A_n = 2 \left[ \frac{(1-x^2)}{\frac{2n+1}{2}\pi} \sin\left(\frac{2n+1}{2}\pi x\right) \right]_0^1 + 2 \cdot \int_0^1 x \frac{\sin\left(\frac{2n+1}{2}\pi x\right)}{\frac{2n+1}{2}\pi} dx$$

$$du = dx \quad u = x$$

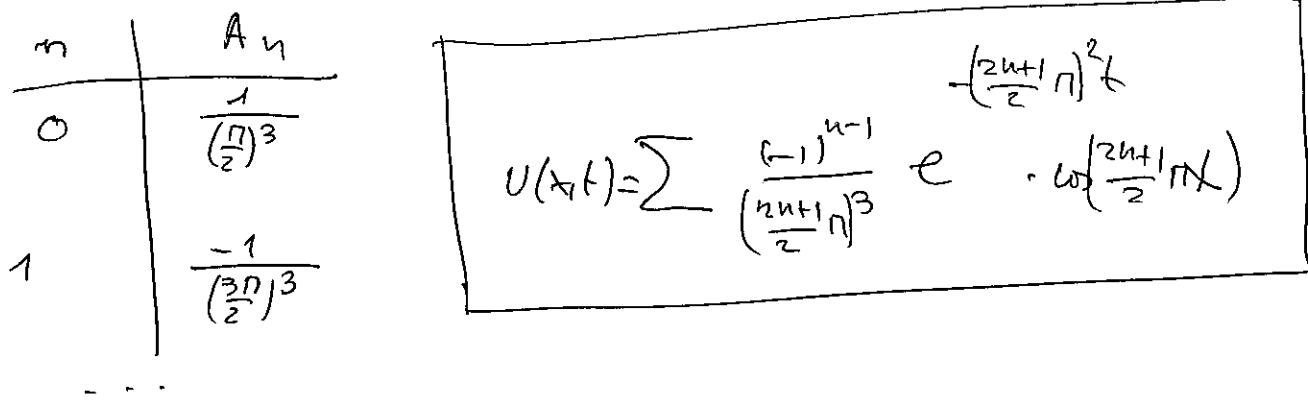
$$dw = \frac{\sin\left(\frac{2n+1}{2}\pi x\right)}{\frac{2n+1}{2}\pi} dx \quad v = -\frac{\cos\left(\frac{2n+1}{2}\pi x\right)}{\left(\frac{2n+1}{2}\pi\right)^2}$$

$$A_n = \frac{1}{4} \left[ \frac{x \cos\left(\frac{2n+1}{2}\pi x\right)}{\left(\frac{2n+1}{2}\pi\right)^2} \right]_0^1 + \frac{1}{\left(\frac{2n+1}{2}\pi\right)^2} \int_0^1 \cos\left(\frac{2n+1}{2}\pi x\right) dx =$$

$$= \left[ \frac{\sin\left(\frac{2n+1}{2}\pi x\right)}{\left(\frac{2n+1}{2}\pi\right)^3} \right]_0^1 = \boxed{\left[ \frac{\sin\left(\frac{2n+1}{2}\pi\right)}{\left(\frac{2n+1}{2}\pi\right)^3} \right]} = \frac{(-1)^{n-1}}{\left(\frac{2n+1}{2}\pi\right)^3}$$

Por tanto la solución es

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-\left(\frac{2n+1}{2}\pi\right)^2 t} \cos\left(\frac{2n+1}{2}\pi x\right)$$



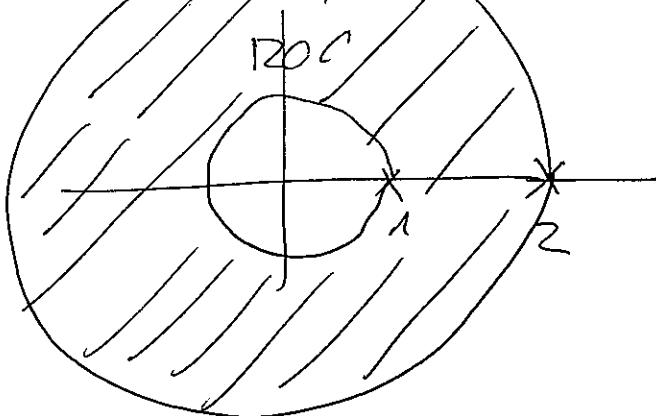
$$\lim_{t \rightarrow 0} u(x,t) = \sum A_n \cos\left(\frac{2n+1}{2}\pi x\right) = u(x,0) \quad \underline{\text{C.I.}}$$

$$\lim_{t \rightarrow \infty} u(x,t) = \sum A_n e^{-\left(\frac{2n+1}{2}\pi\right)^2 t} \cos\left(\frac{2n+1}{2}\pi x\right) = 0$$

Pues un extremo 0 es no válido 2  
por ahí se escapa al calor #

(4)

3) a)  $f(z) = \frac{z^2 - 3}{(z-1)(z-2)}$  ROC:  $1 < |z| < 2$



$$\frac{z^2 - 3}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} = \frac{1}{z-1} + \frac{1}{z-2}$$

$$\boxed{\frac{1}{z-1}} \quad \boxed{\frac{1/z}{1-\frac{1}{z}}} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = \sum_{n=1}^{\infty} \left(\frac{1}{z}\right)^n =$$

$$1 < |z| \quad \boxed{\sum_{n=-\infty}^{\infty} \frac{-1}{z^{n+1}} z^n}$$

$$\boxed{\frac{1}{z-2}} = \boxed{\frac{1/z}{\frac{z}{2} - 1}} = \boxed{\frac{-1/z}{1 - \frac{z}{2}}} = \frac{-1}{z} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n =$$

$$|z| < 2 \quad \boxed{\sum_{n=0}^{\infty} \frac{-1}{z^{n+1}} z^n}$$

Por tanto 
$$\boxed{f(z) = \sum_{n=-\infty}^{-1} z^n - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} z^n}$$

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$$\textcircled{3} \text{ b) } f(z) = \cos\left(\frac{1}{z-1}\right) + \frac{z-\pi}{\sin z}$$

$$\cos z = 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \dots \Rightarrow$$

$$\Rightarrow \cos\left(\frac{1}{z-1}\right) = 1 - \frac{1}{z(z-1)^2} + \frac{1}{4(z-1)^4} - \dots$$

Y por tanto  $\cos\left(\frac{1}{z-1}\right)$  tiene una s. esencial en  $z=1$ .

$$\frac{z-\pi}{\sin z} ? \quad \sin z = 0 \Rightarrow z = 0 + n\pi \text{ con } n \in \mathbb{Z}$$

Para  $n=1$   $z=\pi$  tambien es l.

$$\text{Como } \lim_{z \rightarrow \pi} \frac{z-\pi}{\sin z} = \lim_{z \rightarrow \pi} \frac{1}{\cos z} = \frac{1}{-1}$$

O/H  
(O/C)

$z=\pi$  es s. esencial de  $\frac{z-\pi}{\sin z}$

Por tanto  $f(z)$  tiene:

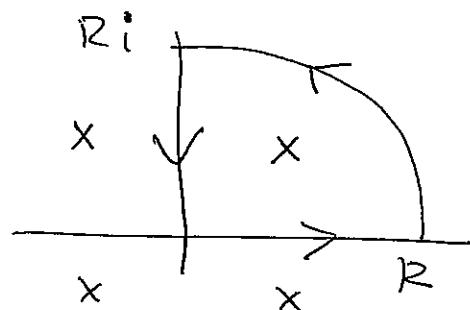
- s. esencial en  $z=\pi$
- polos simples en  $z=n\pi$  con  $n \in \mathbb{Z}, n \neq 1$
- s. esencial en  $z=1$ .

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$$\textcircled{4} \quad a) \quad \int_{-\infty}^{+\infty} \frac{t^2}{t^4+1} dt = \frac{\pi}{\sqrt{2}}$$

Integración en  $\Gamma = [0, R] \cup \Gamma_R \cup [Ri, 0]$

$$z^4 + 1 = 0 \rightarrow z = \sqrt[4]{-1} = \left\{ e^{\frac{\pi}{4}i + \frac{2k\pi i}{4}} \right\}_{k=0,1,2,3}$$



$$\frac{t^2}{t^4+1} \text{ es par} \Rightarrow$$

$$\int_{-\infty}^{+\infty} \frac{t^2 dt}{t^4+1} = 2 \int_0^{\infty} \frac{t^2 dt}{t^4+1} = 2I$$

$$\int_{[0,R]} \xrightarrow[R \rightarrow \infty]{} I$$

$$\text{Lemos } \lim_{z \rightarrow \infty} z \cdot \frac{z^2}{z^4+1} = 0 \stackrel{\text{Lemma 1}}{\Rightarrow} \int_{\Gamma_R} \xrightarrow[R \rightarrow \infty]{} 0$$

$$\text{Para } z \in [Ri, 0] \quad z = x \cdot e^{\frac{\pi}{2}i} ; dz = e^{\frac{\pi}{2}i} dx$$

$$\int_{[Ri, 0]} = \int_{Ri}^0 \frac{x^2 \cdot e^{\frac{\pi}{2}i} \cdot e^{\frac{\pi}{2}i} dx}{x^4 + 1} = i \int_0^R \frac{x^2 dx}{x^4 + 1} = i \cdot I$$

$$\text{Por tanto} \quad \int_I = (1+i)I$$

$$\text{Además} \quad \int_I = 2\pi i \operatorname{Res}(f, \frac{\pi}{2}i)$$

$$\begin{aligned} \operatorname{Res}(f, i) &= \lim_{z \rightarrow e^{\frac{\pi}{2}i}} \frac{(z - e^{\frac{\pi}{2}i}) \cdot z^2}{z^4 + 1} \stackrel{z^4+1}{=} e^{\frac{\pi}{2}i} \lim_{z \rightarrow e^{\frac{\pi}{2}i}} \frac{1}{4z^3} = \\ &= i \cdot \frac{e^{\frac{\pi}{2}i}}{-4} \end{aligned}$$

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Por tanto:

$$(1+i) I = 2\pi i \cdot i \cdot \frac{e^{\frac{\pi}{4}i}}{-4}$$

$$I = \frac{\pi}{2} \cdot \frac{e^{\frac{\pi}{4}i}}{i - e^{\frac{\pi}{4}i}} = \frac{\pi}{2\sqrt{2}}$$

La integral pedida es por tanto:

$$\boxed{\int_{-\infty}^{+\infty} = z I = \frac{\pi}{\sqrt{2}}}$$

4 b)  $f(z) = \sum_{n=0}^{\infty} \frac{n^3}{3^n} z^n$

$$\int \frac{f(z) \cdot z^n z}{z^2} dz$$

$$|z|=1$$

$f(z)$  está escrita como s. de potencias en  $z=0$ .

Además su radio de convergencia  $\gamma$ :

$$R = \left( \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^3}{3^n}} \right)^{-1} = \left( \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n^3}}{3} \right)^{-1} = 3$$

Por tanto  $f \in \mathcal{H}(B(0, 3))$ .  $f(0) = 0$

Por el teorema de Cauchy para los derivados

$$I = 2\pi i \left( f(z) \cdot z^n z \right) \Big|_{z=0} =$$

$$= 2\pi i (f'(0) \cdot z^n z + f(0) \cdot \cos 0) = 2\pi i f'(0) = 0 \quad \#$$

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$$5) \quad Z_u[x[n]] = \sum_{n=0}^{\infty} x[n] \cdot z^{-n}$$

$$\begin{aligned}
 Z_u[y[n]] &= \sum_{n=0}^{\infty} y[n] \cdot z^{-n} = \sum_{n=0}^{\infty} x[n-2] \cdot z^{-n} = \\
 &\quad \text{def. } y[n] \\
 &= x[-2] \cdot z^0 + x[-1] \cdot z^{-1} + \sum_{n=2}^{\infty} x[n-2] z^{-n} = \\
 &= x[-2] + x[-1] z^{-1} + \sum_{n=0}^{\infty} x[n] z^{-n-2} = \\
 &= \boxed{x[-2] + x[-1] z^{-1} + X(z) \cdot z^{-2}} \quad \#
 \end{aligned}$$

## Problemas adicionales :

① Sea  $f$  def. en  $\Omega$  ab<sup>a</sup> convexo de  $\mathbb{R}^2$  tal que  $f$  es armónica en  $\Omega$  y continua en  $\Omega$ . Si  $f$  no es constante, entonces  $f$  alcanza el máximo (y el mínimo) en  $\text{Fr}(\Omega)$ .

Problema de Dirichlet:

Sea  $\Omega$  ab<sup>a</sup> convexo de  $\mathbb{R}^2$

y consideremos el problema:

$$\begin{cases} \Delta u = 0 \\ u|_{\text{Fr}(\Omega)} = f \end{cases} \quad (P)$$

Entonces el problema (P) tiene como mucho una solución.

Sean  $u_1, u_2$  dos soluciones de (P) y

consideremos  $w = u_1 - u_2$ .

$w$  es solución del problema  $(P_H)$ :

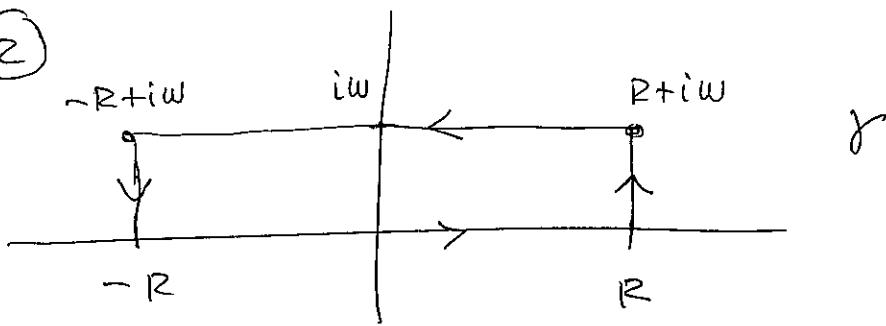
$$\begin{cases} \Delta w = 0 \\ w|_{\text{Fr}(\Omega)} = 0 \end{cases} \quad (P_H)$$

Como  $w|_{\text{Fr}(\Omega)} = 0$  y  $w$  es armónica, por el principio del máximo  $w \equiv 0$  en  $\Omega$  e. d.

$w = u_1 - u_2 = 0$  en  $\Omega$  y por tanto  $u_1 = u_2$  en  $\Omega$ . #

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2



$$\int_{\gamma} e^{\frac{-1}{2}z^2} dz = \int_{[-R, R]} + \int_{[R, R+iw]} + \int_{[R+iw, -R+iw]} + \int_{[-R+iw, -R]}$$

$\Re z \in [-R, R]$ ,  $z = x$

$$\int_{[-R, R]} = \int_{-R}^R e^{\frac{-1}{2}x^2} dx \xrightarrow{R \rightarrow \infty} \int_{-\infty}^{+\infty} e^{\frac{x^2}{2}} dx \stackrel{\text{gato}}{=} \sqrt{2\pi} \quad \boxed{A}$$

$\Re z \in [R, R+iw]$   $z = R+xi$  con  $x \in [0, w]$

$$dz = idx$$

$$\int_{[R, R+iw]} = i \int_0^w e^{\frac{-1}{2}(R+xi)^2} dx = i \int_0^w e^{\frac{-(R^2-x^2)}{2}-xRi} dx \quad \boxed{B}$$

$\Re z \in [-R+iw, -R]$   $z = -R+xi$  con  $x \in [w, 0]$

$$\int_{[-R+iw, -R]} = i \int_w^0 e^{\frac{-1}{2}(-R+xi)^2} dx = -i \int_0^w e^{\frac{-R^2-x^2}{2}-xRi} dx \quad \boxed{C}$$

$$(-R+xi)^2 = (R-xi)^2 = R^2+x^2-2xRi$$

$$(R+xi)^2 = R^2+x^2+2xRi$$

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Si  $z \in [R+i\omega, -R+i\omega]$ ,  $z = x + i\omega$  con  $x \in [R, -R]$

$$\int_{[R+i\omega, -R+i\omega]} = \int_R^{-R} e^{-\frac{1}{2}(x+i\omega)^2} dx = e^{-\frac{x^2}{2} + \frac{\omega^2}{2} - x\omega i} dx$$

$$(x+i\omega)^2 = x^2 - \omega^2 + 2x\omega i$$

$$\rightarrow -e^{\frac{\omega^2}{2}} \int_{-\infty}^{+\infty} e^{\frac{-x^2}{2} - x\omega i} dx \quad \boxed{D}$$

Sumando todas las integrales de la curva:  
pues no hay polos interior a la curva:

$$\begin{aligned} \boxed{B} + \boxed{C} &= i \int_0^w e^{\frac{-R^2+x^2}{2} - xRi} - e^{\frac{-R^2-x^2}{2} - xRi} dx \\ &= e^{\frac{-R^2}{2}} i \int_0^w \end{aligned}$$

$$\boxed{B+C} \xrightarrow[R \rightarrow 0]{} 0$$

Por tanto  $\int \rightarrow \sqrt{2\pi} + e^{\frac{\omega^2}{2}} I = 0$

$$\Rightarrow \boxed{I = +\sqrt{2\pi} e^{-\frac{\omega^2}{2}}} \quad \checkmark$$

Por def. de transformada de Fourier,

tenemos  $\int_{-\infty}^{+\infty} e^{-t^2/2} \cdot e^{-j\omega t} dt = \sqrt{2\pi} e^{-\frac{\omega^2}{2}}$

se tiene que  $\mathcal{F}\left[e^{-t^2/2}\right] = \sqrt{2\pi} e^{-\omega^2/2} \quad \#$